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The Connotations and Forms of Mathematical Understanding

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Abstract: **Mathematical understanding is a critical concern in mathematics education and instruction. It holds a fundamental role in mathematical activities, and all mathematics teaching activities must aim to achieve mathematical understanding. To attain mathematical understanding, we must clarify its connotations and forms. Mathematical understanding is a form of thinking, a process that involves recognizing the essential characteristics of mathematical objects based on existing knowledge and experience. Mathematical understanding manifests through two primary forms: intuition and abstraction.**

Keywords: **Mathematical Understanding; Mathematical Intuition; Mathematical Abstraction.**

I. INTRODUCTION

To sustain competitiveness in the 21st century, educational reform emerged as a prominent trend at the dawn of the century and continues to exert influence today. The reformation of the mathematics curriculum has positioned mathematical understanding as a central research focus in mathematics education. Mathematical understanding pervades the entirety of the mathematics learning process. Without mathematical understanding, meaningful engagement with mathematics is unattainable. It holds a fundamental role in mathematics teaching and learning activities, representing the foremost issue that must be addressed. As James Hibert and Thomas P. Carpenter stated, "One of the most widely accepted ideas in mathematics education theory and practice is that students should understand mathematics."[1] The concept of understanding has become ubiquitous in mathematics curricula and teaching activities. However, it is commonplace for teachers to express frustration, stating, "Haven't I explained this already? Why is it still misunderstood?" In reality, within mathematics instruction, mathematical understanding is frequently conflated with the ability to solve problems, engage in logical reasoning, and perform calculations. Without a clear delineation of the connotations and forms of understanding and mathematical understanding, it becomes challenging to ascertain the attainment of genuine understanding within mathematics instruction.

II. UNDERSTANDING AND MATHEMATICAL UNDERSTANDING

Understanding mathematics, as a primary concern in education, centers on the connotations and forms of understanding. Understanding is not merely the simple acceptance of information but involves a profound mastery and application of knowledge. Specifically, in the field of mathematics, understanding entails a comprehensive cognition and internalization of mathematical concepts, principles, and methods. Based on this, exploring the essence and forms of mathematical understanding is of significant importance for improving mathematical education methods and enhancing teaching quality. Thereafter, we will delve into the basic meaning of understanding and further investigate the specific manifestations of mathematical understanding in educational practice, emphasizing its central role in students' mathematics learning.

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A. The Definition of Understanding

Understanding (理解) represents a fundamental concern in mathematics teaching, with all instructional activities directed towards fostering student comprehension. However, defining understanding poses a significant challenge. Despite teachers employing various strategies to facilitate student comprehension, articulating the essence of understanding remains problematic. Halford G. contends that defining understanding in a universally satisfying manner is inherently difficult.[2] *The Shuowen Jiezi* (*The Explanation of Script and Elucidation of Characters*) elucidates "理" (li) as "治玉也", meaning to process jade, specifically to carve jade from a rough stone. Duan Yucai's annotation of the Shuowen Jiezi states: Unprocessed jade is called "璞" (pu), and "理" (li) refers to the analysis and carving of jade along its natural lines. The general idea is that although jade is hard, it has certain patterns, and it is easier to carve jade from the stone by adhering to these patterns. By extension, "理" (li) means order and reason, reflecting the intrinsic order of objective phenomena. The Shuowen Jiezi elucidates "解" (jie) as "判也", which signifies separation with a knife, such as separating the horn of a cow; another interpretation suggests its original meaning is to dissect an animal. *The Modern Chinese Dictionary* defines understanding as "懂" (dong) and "了解" (liaojie), meaning to know and comprehend.[3] *The Cihai* (*The Sea of Words*) defines it as the process of discerning new phenomena through uncovering relational dynamics. In the English linguistic context, both "understand" and "comprehend" denote understanding. "Under-" comes from a variant of "inter-", meaning within, and "stand" meaning to stand, together implying an alignment in perspective, extended to mean understanding and comprehension. "Comprehend" can be translated as understanding, encompassing, and consisting of; "com-" means all, and "-prehend" means to grasp. Although both words imply understanding, there is a distinction: "understand" emphasizes the acquisition of knowledge post-comprehension, while "comprehend" emphasizes the process of attaining comprehension. Whether from the intrinsic meaning within the English context or the definitions in *the Modern Chinese Dictionary* or *the Cihai*, grasping the connotation of the term "understanding" remains elusive. In the field of curriculum teaching, understanding and knowledge acquisition are distinct. However, from the English context, understanding can be delineated into two distinct meanings: one is the outcome of comprehension, and the other is the cognitive journey to attain it.

In psychology, understanding is regarded as a form of cognition. "Seeing," "hearing," "knowing," and "understanding" are distinct cognitive processes; the former pertains to perception, while the latter pertains to cognition.[4] Naturally, different psychological paradigms offer varied interpretations of the concept of understanding. Behaviorism conceptualizes learning as a stimulus-response association. In school learning, the stimulus unequivocally denotes various forms of knowledge. In this sense, Wiggins, G. & McTighe, J., assert that understanding involves the acquisition of knowledge and skills.[5] How can the acquisition of knowledge and skills be evaluated? When students effectively apply knowledge and skills to novel contexts, it can be inferred that they have achieved genuine understanding. Thus, understanding can be viewed as the capability to apply knowledge in problem-solving.[6] Unlike behaviorism, cognitive psychology posits that learners are active agents, not passive recipients of information. They explain learning as the process wherein new knowledge is assimilated and accommodated through existing knowledge, thereby continuously refining cognitive structures. Ausubel believes that understanding entails integrating new information into the existing cognitive framework, facilitating meaningful assimilation of new and prior knowledge.[7] Within cognitive psychology, "understanding is essentially the process whereby learners construct internal mental representations based on information transmission and encoding, thereby deriving psychological meaning."[8] Mayer, within the cognitive framework, posits that individual understanding occurs in three stages: information selection and input, information encoding, and the reconstruction and integration of representations. Certain cognitive scholars conceptualize understanding as a capability; however, unlike behaviorism, which views understanding as the ability to use knowledge for problem-solving, cognitivism regards it as the ability to represent concepts and contexts.[9] Constructivism asserts that learning is an active process in which individuals construct knowledge grounded in their personal experiences. "Learning is synonymous with understanding, which is a process of attributing meaning."[10] Constructivism contends that knowledge does not exist as an external entity. Learning is predicated on existing knowledge and experience, while external information transcends prior experience, necessitating construction rather than extraction of knowledge. From the constructivist perspective, learning and understanding are equivalent. The learning process involves interpreting the learning object through the individual's existing knowledge and experience, establishing substantive connections between new learning materials and prior knowledge and experience, thereby acquiring new knowledge, constituting the process of understanding. Behaviorism,

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cognitivism, and constructivism offer divergent interpretations of the connotation of understanding. The conceptualizations of understanding can be summarized into two dimensions: one describes the process, referring to the outcome of learning activities; the other considers it a capability, emphasizing the learning process.

B. The Connotation of Mathematical Understanding

All mathematics teaching activities presuppose students' capability to understand; therefore, mathematical understanding emerges as the central issue in mathematics education and instruction. Mathematics curriculum experts, teachers, and mathematics education researchers uniformly regard mathematical understanding as critically important. In China's compulsory education stage, the mathematics curriculum delineates its objectives using the terms "know", "understand", "master" and "apply". When designing teaching plans, mathematics teachers invariably frame teaching objectives from the perspective of mathematical understanding, necessitating comprehension of relevant content throughout the teaching process. What constitutes mathematical understanding? A review of descriptions in mathematics syllabuses or curriculum standards across different periods may offer insights. *The Middle School Mathematics Teaching Syllabus (Revised Draft)* published in 1954 indicated in its instructions: During teaching, avoid overburdening students' memory with an excessive number of mathematical formulas and rules, as this impedes their understanding of the interrelationships and applications of these concepts.[11] Although it does not explicitly define mathematical understanding, it distinguishes understanding from memory, emphasizing that extensive training does not facilitate comprehension of mathematical content. *The Full-time Middle School Mathematics Teaching Syllabus (Draft)* implemented in 1963 asserted: "Correct understanding of mathematical concepts is a prerequisite for mastering basic mathematical knowledge... To facilitate correct understanding, instruction should begin with actual examples comprehensible to students or from prior knowledge, followed by illustrative examples."[12] Similarly, this syllabus does not define understanding explicitly, but it elucidates how to facilitate students' understanding of mathematics by starting with actual or pre-existing knowledge, specifically the students' existing knowledge and experience. The later implemented *Full-time Six-year Key Middle School Mathematics Teaching Syllabus (Draft for Comments)* emphasized instruction according to cognitive principles and proposed four levels of understanding: knowing, understanding, mastering, and flexible application for specific content.[13] For instance, the teaching requirements for plane geometry include knowing the concept of locus, understanding the concepts related to straight lines and circles, and mastering the basic constructions using a compass and ruler. *The Full-time Middle School Mathematics Teaching Syllabus* implemented in 1986 stated: "Students should acquire fundamental knowledge and skills. Initially, students must correctly understand mathematical concepts. New concepts should be introduced through actual examples and existing knowledge. For easily confused concepts, students should be guided to discern the differences and connections between them through comparison. Students should make judgments and reasonings based on a correct understanding of mathematical concepts, thereby grasping mathematical principles and methods; through practice, they should master knowledge and skills and apply them flexibly."[14] This essentially describes the process of mathematics teaching: first, comprehend mathematical concepts, then make judgments and reasonings, and ultimately achieve flexible application. It underscores the fundamental importance of understanding mathematical concepts, highlighting that concepts should be recognized based on students' existing knowledge and experience, and emphasizes that comparison aids in understanding easily confused mathematical concepts. *The Full-time Compulsory Education Mathematics Curriculum Standards (Experimental Draft)* published in 2001 defined understanding as a curriculum goal: the ability to describe the characteristics and origins of objects and clearly explain the differences and connections between objects.[15] This marked the first instance of defining understanding within the mathematics curriculum syllabus and standards, and the revised 2011 compulsory education mathematics curriculum standards retained this definition.

Within the instructional process, mathematical understanding constitutes the primary objective of mathematics teaching. Even if this objective is not fully realized in practice, it remains an ideal pursuit. The objective of mathematical understanding possesses a degree of permanence within the mathematics curriculum and instructional process, endowing the pursuit with enduring significance and value.[16] Mathematical understanding represents the perpetual pursuit within the mathematics classroom and the broader field of mathematics education research. Although distinct schools of thought such as behaviorism, cognitivism, or constructivism have not fully emerged within mathematics education, the field remains profoundly influenced by these paradigms. Hibert and Carpenter assert that "a mathematical concept, method, or fact is understood if it integrates into an individual's internal knowledge network. More precisely, mathematics is understood if its internal representation integrates into this network".[17] This definition is rooted in cognitivism, a view similarly held by Li Shiqi.[18] Mathematical concepts, facts, and methods constitute specific content areas within mathematics. From a cognitivist perspective, the organization of an effective cognitive structure during the process of learning mathematics, wherein mathematical concepts, principles, and rules integrate into the individual's internal knowledge network, signifies understanding. Zhang Wenhui and Wang Guangming believe that mathematical understanding is shorthand for mathematical cognitive understanding. Therefore, they define mathematical understanding as the comprehension of mathematical knowledge during the learning process. "Mathematical understanding refers to the thinking process in which students build appropriate mental representations of knowledge based on existing mathematical knowledge and experience, gradually forming a rich, coherent knowledge network."[19] Chen Qiong and Weng Kaiqing hold the same view. They argue that mathematical understanding is a cognitive activity wherein learners, grounded in their existing knowledge and experience, reinterpret and reconstruct the meaning of textbook content or instructional material, correctly integrating new learning content into the existing cognitive structure, or reorganizing and expanding the original cognitive framework, making the new learning content an integral part of the entire structure, thereby gradually comprehending its essence and principles.[20] Although they all concur that mathematical understanding involves the process of constructing mental representations and forming a knowledge network, Hibert and Carpenter's interpretation of the connotation of mathematical understanding differs. The latter contends that mathematical understanding entails the construction process grounded in the student's existing cognition, constituting a cognitive process. Liu Qingbin and Wang Jingxin believe that mathematical understanding is the process of using existing knowledge and experience to recognize the essence and laws of pure quantities and spatial forms in mathematical objects through thinking.[21] This definition continues to emphasize the learner's existing knowledge and experience, given that mathematics is the study of spatial forms and quantities. Essentially, the objects of mathematics are spatial forms and quantitative relationships; thus, this definition underscores that the object of understanding is mathematical knowledge.

Summarizing the aforementioned understandings and research on mathematical understanding, this paper asserts that mathematical understanding is a cognitive process involving the recognition of the essential characteristics of mathematical objects, grounded in existing knowledge and experience. This definition elucidates the following points: Firstly, mathematical understanding is a cognitive process, constituting a higher-order cognitive activity; secondly, the cognitive activity of mathematical understanding is predicated on existing knowledge and experience, utilizing students' prior knowledge and experience as its foundation; thirdly, the objects of understanding are quantitative relationships and spatial forms; fourthly, it delineates the conditions for achieving mathematical understanding, namely the mastery of the essential characteristics of mathematical objects.

III. BASIC FORMS OF MATHEMATICAL UNDERSTANDING

Mathematical understanding constitutes a cognitive process involving the recognition of the essential characteristics of mathematical objects. Mathematics is the discipline concerned with the study of quantitative relationships and spatial forms. Learning mathematics requires accurately understanding the quantitative relationships and spatial forms present in the real world. To achieve the objective of understanding, comprehending the forms of cognition is essential. Given the staged nature of the cognitive process, the attainment of any cognitive goal is not immediate. Philosophical perspectives on the cognitive process encompass four stages: experience, understanding, judgment, and decision.[22] The mathematics curriculum standards stipulate that"Students should be allotted sufficient time and space to engage in processes of observation, experimentation, conjecture, calculation, reasoning, and verification".[23] These are cognitive activities; observation, conjecture, experimentation, abstraction, and reasoning constitute the principles of cognition in mathematical activities. Cognition adheres to specific principles, representing the forms of cognition. Intuition and abstraction are the fundamental forms of mathematical understanding.

A. The Intuitive Form of Mathematical Understanding

The premise that mathematical understanding is grounded in existing experience is widely accepted in mathematics education. Mathematical understanding that relies on experience predominantly adopts intuitive principles, forming the intuitive approach to mathematical understanding. "Intuition involves the direct perception and recognition of relationships between entities, derived through experience, observation, experimentation, or analogy."[24] The subjects of mathematics comprise quantitative relationships and spatial forms. The intuitive form of mathematical understanding encompasses the direct perception of these quantitative relationships and spatial forms through observation,

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experimentation, or analogy. The intuitive form reflects the individual attributes and external characteristics of the cognitive objects, serving as a direct representation of these attributes. Intuition must be supported by experience; thus, when philosophy views the cognitive process as staged, the experience stage is primary. Hume posited that experience forms the basis of understanding and reasoning, primarily originating from direct experience. The initial stage of mathematical understanding necessitates reliance on intuition; some scholars contend that all mathematical understanding is grounded in intuition. M. Kline asserted that "mathematics relies not on logic but on correct intuition".[25] Locke categorized knowledge into intuitive and rational knowledge, asserting that the reliability of all knowledge hinges on intuition. According to Locke, even though mathematical knowledge can be categorized into various types, modern cognitive psychology divides knowledge into declarative and procedural knowledge. Nevertheless, their comprehension must rely on intuition, with mathematical knowledge being no exception. Indeed, throughout the process of learning and understanding mathematics in primary and secondary schools, a significant amount of mathematical knowledge is conveyed intuitively to facilitate student comprehension. Furthermore, mathematicians' comprehension of mathematics frequently rests on intuitive judgment. Stanislas Dehaene, in *The Origins of Mathematical Intuition*, posits that mathematicians frequently evoke internal "intuition" to solve problems expeditiously and autonomously. "Intuition, as the original and direct presentation of the object, underpins all thinking."[26] Thus, whether concerning the learning of primary and secondary school students or the research conducted by mathematicians, intuition constitutes the fundamental form of mathematical understanding.

Intuition can be classified in various ways from different perspectives. Understanding these classifications aids in gaining a deeper comprehension of the concept of intuition. Philosopher Kant posited that all thinking, whether direct or indirect, ultimately relates to intuition. He categorized intuition into two types: empirical intuition and pure intuition.[27] Traditional epistemology posits that intuition can only be individual, meaning the direct perception of specific real objects by the senses. Philosopher Husserl deemed this perspective too narrow and proposed the existence of general intuition alongside individual intuition.[28] According to Husserl, general intuition, also known as essential intuition, allows individuals to perceive not only the specific characteristics of individual things but also the relationships and commonalities among them, which constitute their essence. Classifying intuition from an educational perspective aims to facilitate teaching. Mathematician Poincaré classified mathematical intuition into three types.[29] First, the intuition of sensations and mental images, referring to the intuitive perception and judgment of shapes and images through direct observation; second, the intuition of induction and experience, which is the perception and judgment formed through experiences, analogies, associations, and imagination; third, the intuition of numbers and mathematical symbols, which is the direct understanding of mathematical objects. Poincaré described intuition from the perspectives of epistemology, methods of mathematical understanding, and objects, categorizing it into three types. In mathematics teaching, real object demonstrations, practical operations, and graphical representations are commonly used to describe objects, facilitating student understanding of mathematical content. Thus, the intuitive forms of mathematical understanding are typically divided into three types: real object intuition, graphical intuition, and operational intuition. Some research further categorizes it into four types: real object intuition, model intuition, graphical intuition, and symbolic intuition.[30] However, even within these four types, different classification forms exist. Kong Fanzhe, Shi Ningzhong, and others posit that its manifestations can be divided into real object intuition, simplified symbolic intuition, graphical intuition, and substitute intuition.[31] Classifying the intuitive forms of mathematical understanding from the perspective of teaching methods facilitates teacher comprehension and acceptance.

The intuitive form permeates the entire process of learning mathematics, playing a crucial role in mathematics learning. M. Kline asserts that mathematical intuition is the direct grasp of concepts and proofs, the primary content of mathematics. Thus, he posits that mathematics relies on correct intuition. Dutch mathematics educator Freudenthal asserted that geometric intuition can indicate what is important, meaningful, and approachable, preventing us from straying into the wilderness of problems, concepts, and methods.[32] In both learning mathematics and in the thinking processes of mathematicians, intuition can inspire solutions and directions for mathematical problems. Compared to the perspectives of Kline and Freudenthal on the role of intuition, mathematician Hilbert's explanation is more straightforward. In his book Intuitive Geometry, he stated: first, diagrams can help depict and describe problems, making them intuitive and simple; second, diagrams can help find ways to solve problems; third, diagrams can help understand and remember the obtained results.[33]

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Additionally, the compulsory education mathematics curriculum standards assert that it helps make complex mathematical problems simple and vivid, and assists in predicting results.[34] Intuitive experiences leave profound impressions, and everything should be placed before the senses as much as possible. Therefore, in mathematics teaching, educators skillfully design lessons to guide students through intuitive forms, helping them discover the shape characteristics of mathematical objects, perceive the quantitative relationships between things, and understand the spatial forms of diagrams.

B. The Abstract Form of Mathematical Understanding

Philosophically, the cognitive process encompasses stages such as experience, thinking, judgment, and decision-making. The mathematical cognitive process involves accumulating mathematical experience and acquiring knowledge. Kant stated: "All human knowledge begins with intuition, proceeds to concepts, and ends with ideas."[35] Concepts, judgments, and reasoning comprise the content of mathematics. For instance, in plane geometry, the content related to quadrilaterals involves defining a parallelogram based on the concept of a quadrilateral. After defining a parallelogram, one must reason whether a quadrilateral with two pairs of equal opposite sides is a parallelogram. The acquisition of mathematical concepts inherently involves mathematical abstraction. *The Modern Chinese Dictionary* defines abstraction as "the process of discarding individual, non-essential attributes from many things to extract common, essential attributes". [36] The English term "abstract" originates from the Latin "abstractio", meaning "a concept or idea not associated with any specific instance", which literally translates to a concept or idea not tied to any particular instance. Thus, in both Chinese and English contexts, abstraction is not linked to specific instances. Mathematics is inherently abstract. Stoljar asserts that modern mathematics abstracts mathematical concepts and axiomatized systems from mathematical material.[37] Xu Lizhi considers mathematical abstraction a cognitive activity involving the extraction of relatively independent aspects, attributes, and relationships from concrete things through this thinking activity.[38] Perception, cognition, and thinking are cognitive activities. Mathematical abstract thinking is a form of rational cognition, reflecting the common, essential attributes of a category of things, indirectly representing their properties. Mathematical understanding is inextricably linked to mathematical abstraction, a fundamental thought form essential for the development of mathematics.

Plato posited that experience is unreliable, arguing that a world of ideas exists where mathematical concepts and propositions reside, accessible only through philosophical contemplation and abstract thinking. In Plato's view, mathematical abstraction pertains to ideas rather than the real world. Aristotle disagreed with Plato's perspective. He posited that the purpose of cognition is to grasp the surrounding things, and abstraction is the recognition of common attributes among a class of things. Aristotle's discussion of abstraction can be summarized as: "General concepts are derived from everyday experiences by abstracting the common properties of many concrete entities; thus, general concepts cannot exist in reality and are manifested in specific things."[39] However, from the perspective of contemporary mathematics, not all mathematical concepts are directly abstracted from the concrete existence of the real world. The debate over the connotation of abstraction has laid the foundation for its classification. Piaget identified two forms of abstraction: empirical abstraction and pseudo-empirical abstraction.[40] The former originates from life experiences and is a direct reflection of objective objects, while the latter is the recognition of common properties of objects, an abstraction of abstractions. Shi Ningzhong suggested that mathematical abstraction can be classified into two types: intuitive description and symbolic representation. This dichotomy aligns with Piaget's classification, as intuitive descriptions in mathematical abstraction directly originate from experience, whereas symbolic representations are abstractions of common properties. Xu Lizhi and others proposed that mathematical abstraction has a hierarchical nature, classifying it into strong abstraction, weak abstraction, and broad abstraction.[41] Other researchers have classified mathematical abstraction based on the nature of the abstracted objects into three types: representational abstraction, principle-based abstraction, and constructive abstraction.[42] Understanding the classification of mathematical abstraction can aid in selecting different forms of abstraction based on the characteristics of mathematical objects in mathematics teaching, thereby helping students comprehend mathematics.

Mathematical abstraction is a fundamental cognitive process essential for the development of the mathematics discipline and the foundation for students to develop core mathematical literacy. The objects of mathematical research are quantitative relationships and spatial forms. Alexandrov stated: "Any form and relationship cannot be completely separated from their content at once; instead, it is through a series of abstractions that mathematical concepts gain independent meaning, becoming the foundation for forming new concepts and conducting new abstractions."[43] This

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illustrates the process of mathematical development, which is not completed in a single abstraction. Concepts abstracted from life experiences form the basis for further abstraction. All mathematical theories are ultimately the result of abstract thinking activities. As Shi Ningzhong stated in *The Abstraction of Mathematics*, mathematics essentially studies abstract entities, which can only be obtained through abstraction. In this sense, abstraction is the fundamental cognitive process on which mathematical development relies. The goal of mathematics instruction is to enable students to grasp the methods for understanding the essence and laws of things. Mathematical abstraction aids students in comprehending mathematical concepts, principles, and reasoning processes, fostering mathematical thinking methods and habits, and ultimately forming rational thinking. The reform of the mathematics curriculum emphasizes developing students' key qualities and abilities, enhancing their core literacy to "observe the world with mathematical perspectives, think about the world with mathematical reasoning, and express the world with mathematical language", thereby laying the foundation for lifelong development.[44] Whether observing with mathematical perspectives, thinking with mathematical reasoning, or expressing with mathematical language, mathematical abstraction is indispensable. Mathematical abstraction is crucial for the development of students' core mathematical literacy.

In conclusion, mathematics profoundly reflects the real world while also being deeply abstracted from everyday life. Mathematical understanding involves students utilizing their existing experiences to recognize the essential characteristics of mathematical objects. These experiences encompass both life and mathematical experiences. The fundamental forms of mathematical understanding are intuition and abstraction. The goal of understanding is to attain a clear comprehension of mathematical concepts, propositions, principles, and reasoning, elevating from perceptual to rational understanding.

IV. CONCLUSION

Mathematical understanding holds a crucial and significant position in mathematics education and teaching. It is not only the core objective of students' mathematical learning but also the foundation of effective mathematics teaching. The connotation of mathematical understanding can be defined as a cognitive process based on existing knowledge and experience, aimed at revealing the essential characteristics of mathematical objects. Through systematic analysis, it is clear that mathematical understanding relies not only on students' existing knowledge and experience but also involves an in-depth recognition of the quantitative relationships and spatial forms of mathematical objects.

The forms of mathematical understanding can be divided into intuitive and abstract forms. The intuitive form directly perceives the external characteristics and specific attributes of mathematical objects through observation, experimentation, and analogy. Intuition plays an important role in the initial stages of mathematics learning, helping students establish a preliminary understanding of mathematical concepts and enhancing their perceptual comprehension of mathematical content through intuitive experiences. The abstract form extracts common and essential attributes from specific mathematical objects and engages in rational thinking through symbolization and conceptualization. The cultivation of abstract thinking is crucial for students to deeply understand mathematical principles and theories.

In the practice of mathematics teaching, teachers need to flexibly employ both intuitive and abstract forms to help students gradually build a comprehensive mathematical cognitive framework. At the initial learning stages, intuitive demonstrations and concrete examples enable students to perceive the characteristics and patterns of mathematical objects directly. As students accumulate mathematical knowledge and enhance their cognitive abilities, they should be gradually guided towards abstract thinking, enabling them to grasp the essential attributes and inherent logic of mathematical objects. Throughout this process, teachers should also encourage students to apply their acquired knowledge to solve practical problems, further deepening their understanding of mathematical concepts and methods.

Moreover, the research on mathematical understanding holds both theoretical and practical significance. By delving into the connotations and forms of mathematical understanding, valuable theoretical support can be provided for mathematics teaching, assisting educators in better grasping instructional content and methodologies, thereby improving teaching outcomes. Simultaneously, research on mathematical understanding can offer insights for mathematics curriculum design and textbook compilation, aligning them more closely with students' cognitive characteristics and learning needs, ultimately promoting the development of students' core mathematical literacy.

In conclusion, mathematical understanding, as the core issue in mathematics education, permeates the entire process of mathematics learning. By clarifying the connotations and forms of mathematical understanding, instructional practice can be better guided, helping students achieve a transition from perceptual to rational understanding, ultimately attaining a profound comprehension of the essential characteristics of mathematical objects. This not only enhances students' mathematical learning capabilities but also lays a solid foundation for their future development. Future research should continue to explore the characteristics of mathematical understanding at different stages and levels, providing richer theoretical and practical bases for the continuous innovation and development of mathematics education.

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